

Accuracy of the Signal-to-Noise Ratio Estimator: A Comment on the Derivation of the Estimator Mean

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The mean of the signal-to-noise ratio estimator used with the symbol synchronizer assembly (SSA) is derived without assuming independence of the sample mean and sample variance errors. The resulting expression is found to differ only slightly from a previous expression determined by assuming independence.

I. Introduction

In Ref. 1, expressions for the mean and variance of the signal-to-noise ratio (SNR) estimator were derived and studied. However, in the derivation of these expressions the assumption was made that the error in the sample mean is independent of the error in the sample variance. This assumption has been the target of some criticism since the errors that produce the error in the sample mean are precisely the same errors that create the error in the sample variance. To answer this criticism, the mean of the SNR estimator was rederived without the assumption of independence of these errors. It was found that the error produced by the independence assumption is negligible for even moderately small sample sizes (in fact, for a sample size of 10, the corresponding error in the estimator mean is bounded by 3.7%).

II. Derivation of the Estimator Mean

We begin by assuming that the reader is familiar with the notation and results of the referenced article. Then the estimated SNR is given by

$$\hat{R} = \frac{\frac{1}{2} \left(\frac{1}{N} \sum_{i=1}^N y_i \right)^2}{\frac{1}{N-1} \sum_{i=1}^N \left(y_i - \frac{1}{N} \sum_{j=1}^N y_j \right)^2} \quad (1)$$

Now assume that $y_i = \mu + \xi_i$ where ξ_i is the error (from the mean μ) of the i th sample. The ξ_i are assumed to be independent, identically distributed, zero-mean random variables having a variance σ^2 . Furthermore, since we desire to apply the Central Limit Theorem (to the sample mean only) we can obtain the same result by assuming

the ξ_i 's are gaussian distributed initially. Thus, the estimated SNR can be expressed as

$$\hat{R} = \frac{\frac{1}{2} \left(\mu + \frac{1}{N} \sum_i \xi_i \right)^2}{\frac{1}{N-1} \sum_i \left(\xi_i - \frac{1}{N} \sum_j \xi_j \right)^2} \quad (2)$$

where the summations are over the N samples. Now by adding and subtracting σ^2 to the denominator and expanding the result in a geometric series we have

$$\hat{R} \approx \frac{1}{2\sigma^2} \left\{ \mu^2(1 - \alpha + \alpha^2) + \frac{2\mu}{N} \sum_i \xi_i (1 - \alpha + \alpha^2) + \frac{1}{N^2} \left(\sum_i \xi_i \right)^2 (1 - \alpha + \alpha^2) \right\} \quad (3)$$

where

$$\alpha = \frac{1}{\sigma^2(N-1)} \sum_i \left(\xi_i - \frac{1}{N} \sum_j \xi_j \right)^2 - 1 \quad (4)$$

Therefore, the mean of the estimator can be expressed as

$$E\{\hat{R}\} \approx \frac{1}{2\sigma^2} [E(T_1) + E(T_2) + E(T_3)] \quad (5)$$

where T_1 , T_2 and T_3 are the three terms in the bracket of Eq. (3). Now consider the terms separately. For T_1 , we have

$$E(T_1) = \mu^2 - \mu^2 E(\alpha) + \mu^2 E(\alpha^2) \quad (6)$$

But note that α is the error of the sample variance. Thus,

$$E(\alpha) = 0$$

$$E(\alpha^2) = \frac{\sigma_{sv}^2}{\sigma^4} \quad (7)$$

where σ_{sv}^2 is the variance of the sample variance. Therefore

$$E(T_1) = \mu^2 + \frac{\mu^2}{\sigma^4} \sigma_{sv}^2 \quad (8)$$

For T_2 , note that all of its terms involve odd powers of the ξ_i 's. Thus, since the odd moments of a zero-mean gaussian random variable vanish we have

$$E(T_2) = 0 \quad (9)$$

To compute the expectation of T_3 we must first expand the α 's. Using the definition of α we have

$$E\{T_3\} = E \left\{ \frac{3}{N^2} \left(\sum_i \xi_i \right)^2 - \frac{3}{N^2 \sigma^2 (N-1)} \left(\sum_i \xi_i \right)^2 \sum_j \xi_j^2 + \frac{3}{N^3 \sigma^2 (N-1)} \left(\sum_i \xi_i \right)^4 + \frac{1}{N^2 \sigma^4 (N-1)^2} \left[\left(\sum_i \xi_i \right)^2 \left(\sum_j \xi_j^2 \right)^2 \right] - \frac{2}{N^3 \sigma^4 (N-1)^2} \left(\sum_i \xi_i \right)^4 \sum_j \xi_j^2 + \frac{1}{N^4 \sigma^4 (N-1)^2} \left(\sum_i \xi_i \right)^6 \right\} \quad (10)$$

To evaluate the expected value of these terms is not difficult in theory but does require a very large amount of algebra. For example, the summation in the first term can be expanded to yield

$$\left(\sum_i \xi_i \right)^2 = \sum_i \xi_i^2 + \sum_i \sum_{j \neq i} \xi_i \xi_j \quad (11)$$

and by using the independence of the ξ_i 's we have

$$E \left\{ \left(\sum_i \xi_i \right)^2 \right\} = N \sigma^2 \quad (12)$$

By similarly expanding the remaining terms and taking the expected value,

$$E\{T_3\} = \frac{1}{N^3(N-1)^2} (N^4 - N^2 + 6N + 6) \sigma^2 \quad (13)$$

Combining the expected values of T_1 , T_2 , and T_3 yields

$$E\{\hat{R}\} \approx \frac{\mu^2}{2\sigma^2} \left[1 + \frac{\sigma_{sv}^2}{\sigma^4} + \frac{\sigma^2}{N\mu^2} + \frac{(N^3 - N^2 + 3N + 3)}{N^2(N-1)} \left(\frac{2}{N-1} \right) \left(\frac{\sigma^2}{N\mu^2} \right) \right] \quad (14)$$

Now, we note that (see Ref. 1)

$$\sigma_M^2 = \frac{1}{N} \sigma^2$$

and from Appendix A of the article we know that for gaussian random variables

$$\sigma_{sv}^2 = \frac{2}{N-1} \sigma^4$$

Thus, Eq. (14) becomes

$$\begin{aligned}
 E\{\hat{R}\} &\approx \frac{\mu^2}{2N\sigma_M^2} \left[1 + \frac{\sigma_{sv}^2}{\sigma^4} + \frac{\sigma_M^2}{\mu^2} + \frac{N^3 - N^2 + 3N + 3}{N^2(N-1)} \left(\frac{\sigma_{sv}^2}{\sigma^4} \right) \left(\frac{\sigma_M^2}{\mu^2} \right) \right] \\
 &= \frac{\mu^2}{2N\sigma_M^2} \left[1 + \frac{\sigma_{sv}^2}{\sigma^4} + \frac{\sigma_M^2}{\mu^2} + k \left(\frac{\sigma_{sv}^2}{\sigma^4} \right) \left(\frac{\sigma_M^2}{\mu^2} \right) \right]
 \end{aligned} \tag{15}$$

where

$$k = \frac{N^3 - N^2 + 3N + 3}{N^2(N-1)}$$

Returning now to Ref. 1 (Eq. (8)) we found that the expression for the estimator mean was

$$E\{\hat{R}\} = \frac{1}{2NA} [1 + A + B + AB] \tag{16}$$

where

$$A = \frac{\sigma_M^2}{\mu^2} \text{ and } B = \frac{\sigma_{sv}^2}{\sigma^4}$$

Equations (15) and (16) are identical except for the factor k multiplying the AB term of Eq. (15). Thus, if k is close to 1, then the original assumption of independence of the sample mean and variance errors is justified. However, note that

$$k = 1 + \frac{3(N+1)}{N^2(N-1)}$$

Thus, k is indeed close to unity for even moderate N . For example, if $N = 10$, then $K = 1.037$, so that the estimator mean is increased by approximately 3.7% if $AB \gg 1 + A + B$ and is increased by a much smaller percentage otherwise.

Reference

1. Lesh, J. R., "Accuracy of the Signal to Noise Ratio Estimator," in *The Deep Space Network Progress Report*, Technical Report 32-1526, Vol. X, pp. 217-235. Jet Propulsion Laboratory, Pasadena, Calif., Aug. 15, 1972.